Buy Signal from Limit Theorem

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Paper submitted to CUSJ 2018 Issue

October 15, 2020

Abstract

This paper studies price movements by limit theorem point of view. We develop a series a buy signals given a trading frequency in the underlying market.

Contents

1 Introduction **2**

2 Theoretical Framework **2**
2.1 Architecture 2
2.2 Theories 2

3 Algorithms **3**

4 Conclusion **4**

5 Appendix **4**
5.1 **Proof of Theorem** 2.2.1 4
5.2 **Proof of Theorem** 2.2.2 4
5.3 **Proof of Theorem** 2.2.3 4
5.4 **Proof of Theorem** 2.2.3 5

1 Introduction

Security prices follow random walk. Although some scholars doubt the concept of efficient market hypothesis, there are fruitful amount of previous research exploring and studying this topic. Some notable papers are by Fama and French [2], [3], [4], and [5]. Other scholars such as Malkiel have also provided persuasive empirical evidence that we do observe data in favor of efficient market hypothesis [6].

2 Theoretical Framework

2.1 Architecture

Definition 2.1.1. For each company i at a time t, we observe a price, that is,

$$p_{i,t} \tag{2.1.1}$$

Definition 2.1.2.

$$\text{SMA}_n = \frac{1}{n}\sum_{i=n}^{t-n} p_{i,t-n} \tag{2.1.2}$$

Definition 2.1.3. Let n be the same value from Definition 2.1.2, denote

$$\text{EMA}_n = (p_{i,t} - \text{EMA}_{n-1}) \times m + \text{EMA}_{n-1} \tag{2.1.3}$$

while $m = \frac{2}{n+1}$.

2.2 Theories

Theorem 2.2.1. *For some n, suppose we have price by Definition 2.1.1 and SMA by Definition 2.1.2, then we have*

$$p_{i,n} - SMA_n \Rightarrow \chi \tag{2.2.1}$$

while χ is the standard normal distribution.

Theorem 2.2.2. *Let the distance between price and moving average to be $\mathbf{D}$ which is defined as*

$$\mathbf{D}_i := p_n - SMA_n$$

while $i = n$, and then we can consider $\mathbf{D}_i$ to be i.i.d. with $\mathbb{E}\mathbf{D}_i = 0$ and $\mathbb{E}\mathbf{D}_i = \sigma^2 \in (0, \infty)$. Then

$$\sum_{m=1}^{n} \mathbf{D}_m \Big/ \left(\sum_{m=1}^{n} \mathbf{D}_m^2\right)^{1/2} \Rightarrow \chi \tag{2.2.2}$$

while χ is the standard normal distribution.

Theorem 2.2.3. *Let the distance between price and moving average to be $\mathbf{D}$ which is defined as*

$$\mathbf{D}_i := p_n - SMA_n$$

while $i = n$, and then we can consider $\mathbf{D}_i$ to be i.i.d. with $\mathbb{E}\mathbf{D}_i = 0$ and $\mathbb{E}\mathbf{D}_i = \sigma^2 \in (0, \infty)$. Let $S_n = D_1 + \cdots + D_n$. Let N_n be a sequence of nonnegative integer-valued random variables and c_n a sequence of integers with $c_n \to \infty$ and $N_n/c_n \to 1$ in probability. Then

$$S_{N_n}/\sigma\sqrt{a_n} \tag{2.2.3}$$

where χ is a standard normal distribution.

Theorem 2.2.4. *Let the distance between price and moving average to be* $\mathbf{D}$ *which is defined as*

$$\mathbf{D}_i := p_n - SMA_n$$

while $i = n$*, and then we can consider* $\mathbf{D}_i$ *to be i.i.d. with* $\mathbb{E}\mathbf{D}_i = 0$ *and* $\mathbb{E}\mathbf{D}_i = \sigma^2 \in (0, \infty)$*. Let* $S_n = D_1 + \cdots + D_n$*. Let* $N_t = \sup\{m : S_m \leq t\}$*. Then as* $t \to \infty$*,*

$$(\mu N_t - t)/(\sigma^2 t/\mu)^{1/2} \Rightarrow \chi \tag{2.2.4}$$

while χ *is the standard normal distribution.*

3 Algorithms

This section we take the theorems above as given and we introduce a series of algorithms targeting buy signals.

Algorithm 3.0.1. Given a buy frequency by an investor c, for all i in a stock pool of companies:

Step 1. Observe price p_t for each company
Step 2. Store $p_{i,t}$
Step 3. Compute SMA_n

$$\mathbf{D}_n := p_{i,n} - \text{SMA}_n$$

Step 4. If $\mathbf{D}_n \leq c$, print "+1"; else, print "0".
Print a collection of "+1" per company i per n.

Algorithm 3.0.2. Given a buy frequency buy an investor c, for all i in a stock pool of companies:

Step 1. Observe price p_t for each company
Step 2. Store $p_{i,t}$
Step 3. Compute $\mathbf{D}_n := p_{i,n} - \text{SMA}_n$

$$\mathbf{Signal}_n := \sum_{m=1}^{n} \mathbf{D}_m \Big/ \left(\sum_{m=1}^{n} \mathbf{D}_m^2\right)^{1/2}$$

Step 4. If $\mathbf{Signal}_n \leq c$, print "+1"; else, print "0".
Print a collection of "+1" per company i per n.

Algorithm 3.0.3. Given a buy frequency buy an investor c, for all i in a stock pool of companies:

Step 1. Observe price p_t for each company
Step 2. Store $p_{i,t}$
Step 3. Compute $S_{N_n} = D_1 + \cdots + D_n$, σ is the variance of D_i,

$$\textbf{Self-Norm}_n := S_{N_n}/\sigma\sqrt{a_n}$$

Step 4. If $\mathbf{Self-Norm}_n \leq c$, print "+1"; else, print "0".
Print a collection of "+1" per company i per n.

Algorithm 3.0.4. Given a buy frequency buy an investor c, for all i in a stock pool of companies:

Step 1. Observe price p_t for each company
Step 2. Store $p_{i,t}$
Step 3. Compute the mean μ and the variance σ,

$$\mathbf{Renewal}_n := (\mu N_t - t)/(\sigma^2 t/\mu)^{1/2} \Rightarrow \chi$$

Step 4. If $\mathbf{Renewal}_n \leq c$, print "+1"; else, print "0".
Print a collection of "+1" per company i per n.

Algorithm 3.0.5. Given results from the above algorithms, that is, **Algorithms** 3.0.1, 3.0.2, 3.0.3, and 3.0.4, run

Step 1. Retreat ($\mathbf{D}_n$), ($\mathbf{Signal}_n$), ($\mathbf{Self\text{-}Norm}_n$), and ($\mathbf{Renewal}_n$).

Step 2. Each i, at any time t, compute

$$\text{Buy} : \sum \text{all signals} := \text{val}(\mathbf{D}_n) + \text{val}(\mathbf{Signal}_n) + \text{val}(\mathbf{Self} - \mathbf{Norm}_n) + \text{val}(\mathbf{Renewal}_n)$$

Step 3. print(t); print(buy).

4 Conclusion

5 Appendix

5.1 Proof of Theorem 2.2.1

This is a relatively easy proof since the definition follow the premises of the Central Limit Theorem. That is, we have $p_{i,n}$ and SMA_n that are i.i.d.. Then by C.L.T., $p_{i,n} - \text{SMA}_n \Rightarrow \chi$ while χ stands for standard normal distribution.

Q.E.D.

5.2 Proof of Theorem 2.2.2

From weak law we know that

$$\sum_{m=1}^{n} D_m^2 / n\sigma^2 \to 1.$$

Also note $y^{-1/2}$ s continuous at 1, then we have

$$\begin{aligned} \left(\sigma^2 n \Big/ \sum_{m=1}^{n} \mathbf{D}_m^2 \right)^{1/2} &\to 1, \text{ in probability, explained in Remark } \star \\ \frac{\sum_{m=1}^{n} \mathbf{D}_m}{\sigma\sqrt{n}} \left(\frac{\sigma^2 n}{\sum_{m=1}^{n} \mathbf{D}_m^2} \right)^{1/2} &\Rightarrow \chi \cdot 1 = \chi, \text{ from the result in Remark } \star \end{aligned}$$

Notice that the $\star$ is because in Weak Convergence, there is a theorem stated that $X_n \Rightarrow X_\infty$ if and only if for every bounded continuous function g we have $\mathbb{E}g(X_n) \to \mathbb{E}g(X_\infty)$. Since we discussed the continuity of function $y^{-1/2}$ at 1, this line is valid.

Q.E.D.

Remark 5.2.1. From [1], Section 2, the theorem stated the following. Suppose $X_n \Rightarrow X$, $Y_n \geq 0$, and $Y_n \Rightarrow c$, where $c > 0$ is a constant, then $X_n Y_n \Rightarrow cX$.

5.3 Proof of Theorem 2.2.3

From Kolmogorov's inequality we know

$$\mathbb{P}\left(\max_{(1-\epsilon)c_n \leq m \leq (1+\epsilon)c_n} |S_m - S_{[(1-\epsilon)c_n]}| \right) \leq 2\epsilon/\delta^2$$

If $D_n = S_{N_n}/\sigma\sqrt{c_n}$ and $Y_n = S_{c_n}/\sigma\sqrt{c_n}$, then it follows that

$$\limsup_{n\to\infty} P(|D_n - Y_n| > \delta) \leq 2\epsilon/\delta^2, \forall\epsilon$$

then we have $P(|D_n - Y_n| > \delta) \to 0$ for each $\delta > 0$, i.e., $X_n - Y_n \to 0$ in probability. This is because of the Cnverging together lemma stated in Weak Convergence part of [1]. We state the theorem in remark below.

Q.E.D.

Remark 5.3.1. Suppose $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, where c is a constant then $X_n + Y_n \Rightarrow X + c$. A useful consequence is that if $X_n \Rightarrow X$ and $Z_n - X_n \Rightarrow 0$ then $X_n \Rightarrow X$.

5.4 Proof of Theorem 2.2.3

From convergence theorem, we know that

$$\frac{N_t}{t\mu} \to 1$$

so from Theorem 2.2.3 we have

$$\frac{S_N - \mu N_t}{\sigma\sqrt{t/\mu}} \to 0$$

then it is sufficient to show $(S_n - t)/\sqrt{t} \to 0$ since it follows that $\frac{(\mu N_t - t)}{\sqrt{\sigma^2 t/\mu}} \Rightarrow \chi$.

We have given finite variance, that is, $\sigma^2 < \infty$, so by D.C.T. (Dominated Convergence Theorem), we have

$$\begin{aligned} P\left(\max_{1\le m\le 2t\mu} Y_m > \epsilon\sqrt{t}\right) &= \tfrac{2t}{\mu} P(Y_1 > \epsilon\sqrt{t}) \\ &= \tfrac{2}{\mu\epsilon^2}\mathbb{E}(Y_1^2; Y_1 > \epsilon\sqrt{t}) \to 0 \end{aligned}$$

which proves that $(S_n - t)/\sqrt{t} \to 0$ is true. Hence, this completes the proof.

Remark 5.4.1. This is because of the Converging together lemma stated Weak Convergence. Please see Remark 5.3.1.

Q.E.D.

References

[1] Durrett, Rick, "Probability: Theory and Examples ", 4E.

[2] Fama, Eugene F. (1992), "The cross-section of expected stock returns," *Journal of Finance*, 47, 427-465.

[3] Fama, Eugene F. (1993), "Common risk factors in the returns on stocks and bonds," *Journal of Financial Economics*, 33, 3-56.

[4] Fama, Eugene F. (1996), "Multifactor portfolio efficiency and multifactor asset pricing," *Journal of Financial and Quantitative Analysis*, 31, 441-465.

[5] Fama, Eugene F. (2006), "Profitability, investment, and average returns,", *Journal of Financial Economics*, 82, 491-518.

[6] Malkiel, Burton G. (2007), "Reflections on the Efficient market Hypothesis: 30 Years Later," *The Financial Review*, 40, 1-9.

[7] Yin, Yiqiao (2016), "Anomaly Correction by Optimal Trading Frequency", *CUSJ*,