# Martingale to Optimal Trading

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October 1, 2016

**Abstract**

This paper presents a theoretical model to systematically bid in the market when the prices drop. The goal for this model is to create a market environment with the same expected return but a lot more smaller standard deviation, hence higher Sharpe Ratio.

# Contents

# 1 Introduction

Security prices are generally understood as random walk. Although some scholars still doubt the concept of efficient market hypothesis, there are scholars like Malkiel shown us empirical evidence that we do observe data in favor of efficient market hypothesis. He has shown us that professional investment managers do not outperform their index benchmarks and has provided us evidence that prices do digest all available information [4].

In this paper, we start with the characteristic that price follows random walk. That is, we assume market is efficient over the long run and security prices do reflect all available information out there. This is also to say that the first order derivative of price (price changes) follows binomial distribution (big probabilities that absolute values of return are small and small probabilities that absolute values of return are big). Moreover, we prove that expected return of security prices is a martingale. Samuelson (1965) advocated that price fluctuated randomly and the movement of prices can be seen as a stochastic process [5].

Jegadeesh and Titman (2007) has found evidence that strategies of buying stocks performed well in the past and selling stocks that performed poorly in the past generate significant positive returns in 3- and 12- month holding period [3]. Similar ideas are presented by De Bondt and Thaler (1985, 1987) [1][2]. This up-minus-down momentum factor in the security market provides another important ground work for us. If the strategies introduced by these scholars are worth running (which proved that they are), there would be rational investors out there who want to try these strategies. This provide liquidities for our model. If no one is willing to buy high sell low, it does not make sense for us to argue to buy low sell high. Hence, we would assume that there is some liquidities out there provided, that is, the market exists.

Another thing that we need to pay attention to is the concept of anomaly. A value strategy can be an anomaly if significant alpha can be found. A momentum strategy can be an anomaly if its alpha is significant. We define anomalies among a pool of expected returns to be the ones cannot be tolerated by an investor subjectively. This is a definition that has not been discussed before. As a matter of fact, this definition is difficult to start any quantitative approach because of the adjective "subjective". We set up this definition on purpose because we will introduce a trading frequency $\delta$ to be associated with the investor's tolerance of the security prices and hence we can start to have some discussions about anomalies.

The questions we want to answer in this paper originate from the assumptions above. If market is efficient and incorporates all publicly available information, it is rational to infer that a market grows by a various amount of factors out there, say GDP, NGDP, interest rates, fundamentals, technicals, algorithms, and etc. since all of the information are public. Then we would ideally see a relatively straightened line of prices throughout the history if we compare to other indicators. However, we observed two very different variations between security market and economic market. Scholars can find factors and use pricing models to find out why the security market is so volatile, hence, they can explain the reason why we have such observations, i.e., there are a lot more spikes in the price chart of stock market than there are in GDP chart (see Figure 1 and 2). Is it all possible to simply buy the market when market is low and just let the market grow with the economy? If so, how do we do it in a consistent and rational sense that the lower the market dips the more we buy?

Figure 1: The return after interest rate with $1 investment in the market from July of 1926 to June of 2016. *Source: From Ken French Data Library.*

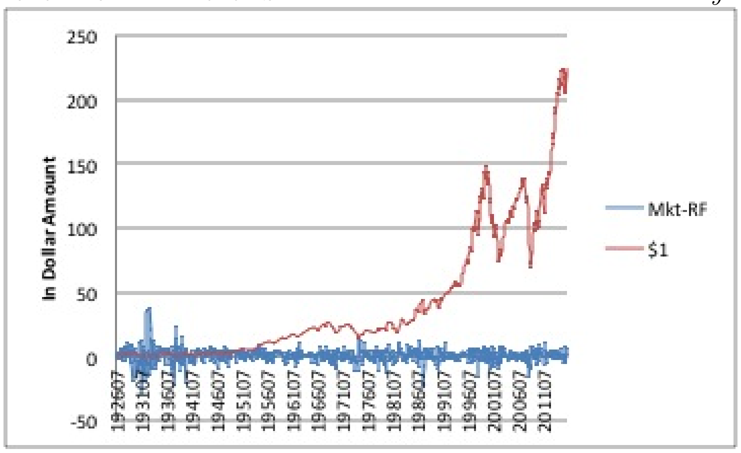
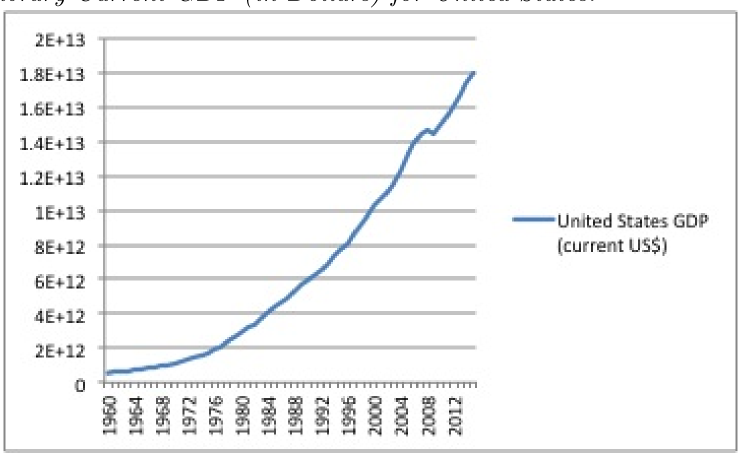



Figure 2: The GDP in US Dollars from 1960 to 2015. *Source: From World Bank Library Current GDP (in Dollars) for United States.*



# 2 Price as a Martingale

At each time node $n$, we have some kind of information about $\omega \in \Omega$ that we know contained in a filtration, $F_n$. As $n$ increases, we incorporate more information about $\omega$, and hence $\omega$ increases. We introduce the following definition.

**Theorem 2.1.** *A discrete-time stochastic process, price, $P = \{P_t, t = 0, 1, ...\}$ is a martingale with filtration $\mathbb{F} = \{F_t, t = 0, 1, ...\} \forall t \in \mathbb{N}^+$. Moreover, the first order derivative of price, $r_t = \frac{\partial P}{\partial t}$ is also a martingale.*

It is easy to check this since one needs to check that the discrete-time stochastic variable price, $P_t$, satisfies all the properties of a martingale (see Appendix). From the definition, we can take the first order derivative of the price and the properties would still be satisfied. We can argue that the the expected return at any time node is equal to the initially selected time node. Furthermore, we argue that the expected return for prices at time node $t$ conditional on the filtration is the same as the one before.

**Theorem 2.2.** *Suppose that $r = \{r_t, t = 0, 1, ...\}$ is a martingale with filtration $\mathbb{F}$. Then*

*(i)* $\mathbb{E}(r_t) = \mathbb{E}(r_0)\ \forall t \in \mathbb{N}^+$

*(ii)* $\mathbb{E}(r_t|F_m) = r_s, \forall r > 0, \forall s > 0,$ *and* $s < t$.

Alternatively, we can also present large dataset to show that years of historical data is plotted and can look like normal distribution. However, since data set has survival bias, we choose a theoretical approach to show you a more completed argument.

With the idea of price following martingale, we can look at the next section to discuss moving averages.

# 3 Mathematical Model

## 3.1 Moving Averages

We look at price as a time-series function. That is, for each time node $t$, we have an executed price $P$. This can be seen as a discrete function with a map from time to price. We can denote price as $P_t$.

Anything can happen any time in the market and an individual price level is usually very volatile. We incorporate the concept of moving average so that we can look at the average price of some period in the past, which allows us to overlook the trading noise in the market.

**Definition 3.1.** Simple moving average (SMA) takes the average of prices traded $n$ time period(s) in the past,

$$SMA_n = \frac{1}{n} \sum_{t=0}^{n} P_t.$$

We can also implement the concept of exponential moving average, which is to add a certain amount of weight to the current prices when calculating the averages.

## 3.2 Price to Moving Averages

Based on the moving averages of the prices, we can establish the concept of price-to-moving averages.

**Definition 3.2.** Price-to-Moving Average is the ratio of price over a selected period of moving averages. For $n$ periods, we have

$$P_t - SMA_n = \frac{P_t}{\frac{1}{n}\sum_{t=0}^{n} P_t} - 1$$

.

We can interpret this definition by our assumptions. We assumed that price follows random walk. The averages of a random walk should also be random walk. That is to say, we have two series of numbers following binomial distributions, which implies that the ratio should also be a binomial distribution. The data had an average expected return of 0.65%. If we imagine this to be the initial x-axis, we would observe a binomial distribution for price-to-moving averages for all three data series.

Figure 3: The graph shows returns after interest rate with $1 investment in the market along with three other moving averages with time periods to be 50-day, 100-day, and 150-day from July of 1926 to June of 2016. *Source: From Ken French Data Library.*

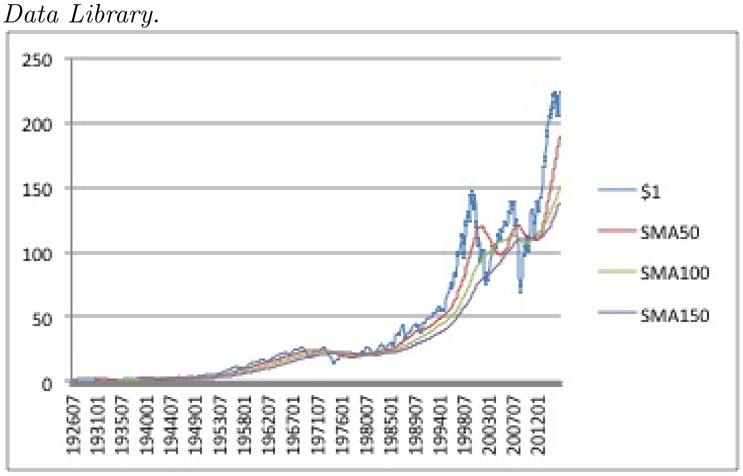
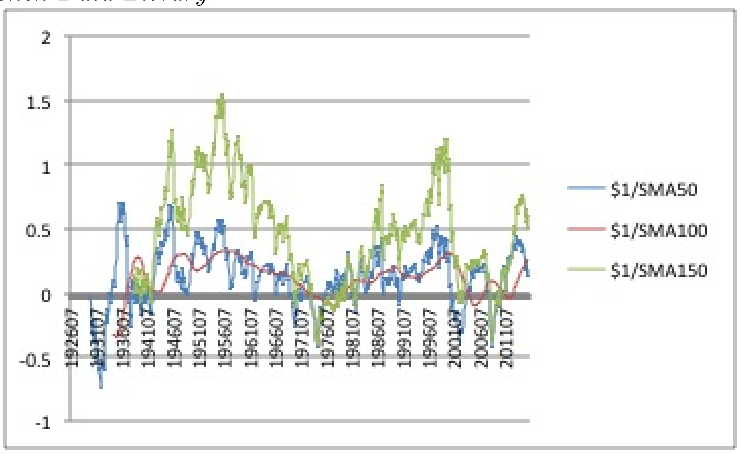


Figure 4: The graph shows price-to-moving averages with time periods to be 50-day, 100-day, and 150-day from July of 1926 to June of 2016. *Source: From Ken French Data Library.*




### 3.3 Trading Frequency

We defined in the introduction section of this paper that we call anomalies to be the ones among a pool of observations that cannot be tolerated by investors subjectively. This concept is associated with a trading frequency $\delta$. An investor could be greedy or moody when he made an investment decision. We do not care. We summarize all of those emotions into $\delta$. Emotionally impatient investors tend to be less educated and would buy very often, hence a high trading frequency ($\delta$ is big). Emotionally patient investors tend to be more educated and would buy once in a while, hence a low trading frequency ($\delta$ is small).

Based on each price-to-moving average, we can calculate standard deviation, $\sigma$. For every $p_t - SMA_n$, we have a calculated $\sigma_{t,n}$. From binomial distribution, we know that the probability that $p_t - SMA_n$ is high happens not that often whereas the probability that $p_t - SMA_n$ is low happens pretty often (i.e. bell curve). Hence, for the observation space $X = (t, n)$ and a particular event $k = (i, j)$, we have

$$\forall i, j \in \mathbb{R}, Pr(p_t - SMA_n = p_i - SMA - j) = Pr(X = k)$$

$$= \binom{n}{k} p^k (1-p)^{n-k} \forall k \in \mathbb{Z}, \ where \ \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The interpretation is the following. There exists some $i$ and $j$ such that the probability of the price-to-moving average equals to this particular $i$ and $j$ is defined by the binomial equation.

The next question we need to think about is how do we understand or even quantify how many particular pairs of $(i, j)$ we need to choose. The answer to this question is dependent on individual investor, which is associated with trading frequency $\delta$. Moreover, the trading frequency

$\delta$ is dependent on standard deviation $\sigma_{t,n}$. In other words, we need the following definition.

**Definition 3.3.** With the standard deviation generated by $(t', n')$ to be greater or equal to that generated by $(t, n)$, the particular event $k$ is the product of $\delta$ and total observations with (buy side) trading probability $\delta$ defined as the following

$$\exists h \in \mathbb{R} \ s.t. \ for \ (t,n) \ and \ (t',n')$$

$$\delta_t = Pr(\sigma_{t,n} \geq h\sigma_{t,n}) = \frac{\sum_{(t',n')} \mathbb{I}_{(t',n')}}{\sum_{(t,n)} \mathbb{I}_{(t,n)}}$$

*with*

$$\mathbb{I}_{(t',n')} = \begin{cases} 1, \ when \ h\sigma_{t',n'} > \sigma_{t,n} \\ 0, \ when \ h\sigma_{t',n'} \leq \sigma_{t,n} \end{cases}, \ (sign \ changes \ for \ sell \ side)$$

*and*

$$\mathbb{I}_{(t,n)} = \begin{cases} 1, \ if \ \exists \sigma_{(t,n)} \\ 0, \ otherwise \end{cases}$$

*while*

$$\sigma_{t,n} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}((p_t - SMA_i) - \frac{1}{n}\sum_{i=1}^{n}(p_t - SMA_i))^2}$$

This equation looks complicated, yet it is easy to understand. The amount of event $k$ happens in a way that is dependent on trading frequency $\delta$. Trading frequency $\delta$ is defined as a probability, which is a probability calculated to be the amount of times standard deviation of price-to-moving averages to be greater or equal than an ideal amount of times of all observations. The indicator, $\mathbb{I}$, is to pick out the amount of times that the calculated standard deviation satisfies condition, which is a coefficient, $h$, by particular investor. The equation says, if the standard deviation wanted is greater than standard deviation of the entire time-series by a factor of $h$, then it is indicated as 1 otherwise 0. The trading probability $\delta$ is then calculated by the ratio of this number over total number that the standard deviation can be calculated (total number of the population of the time-series).

In other words, we are looking at areas in the binomial distribution that are either too high or too low (far away from the mean).

## 3.4 Optimal Trading Price

How often do we have to trade? Each of us has a unique risk tolerance as well as expected return for our investment. Then each of us has a particular $k$ we are using when making investment decisions. This is to say, we need to find the right amount of $k$ around the price when the price-to-moving average is the lowest. The bigger $k$ an investor has the earlier he needs to start buying.

The first goal is to find the optimal price-to-moving averages level. We need to take the first order derivative of price-to-moving averages and set the result to zero. We have the following theorem.

**Theorem 3.4.** *$\forall n$ and $\forall \triangle > 0$ no matter how small, when $\sum_{t=0}^{n} P_t = \sum_{t=0}^{n} P_{t+\triangle}$, we have a critical price to act on it.*

The interpretation is fairly intuitive. We look at a series of price and the price goes up or down. We calculate the average of prices in a certain period. Then we have a series of average prices. We look at the ratio of price over moving averages and we treat the ratio to be a new number series with some arbitrary mean. We calculate the sum of prices in the past certain of time nodes. The optimal price level occurs at a time when the sum of the averages at two time nodes does not change at all (or change tiny little). Intuitively speaking, this is a time when the price is going down too much and now it is starting to bounce back, causing the sum of the past prices to be relatively stable. For example, we calculate the sum of hundreds of prices in the past. If the price keeps going down, we are supposed to get a sum smaller and smaller, yet one day the sum stays the same all of the sudden. Then this price can be critical to buy and vice versa. The complete proof is shown in the Appendix 4.1.

In practice, we will probably find some price areas as critical price level. That is to say, we would not find a perfect equivalent sums of prices in the data. We would probably see a range of prices that satisfy the condition. Then we can apply trading frequency $\delta$. The bigger the $\delta$ is the bigger the time node is. Therefore, a consistent investor who is true to his own preference would start to buy earlier and vice versa (sell earlier). The smaller the $\delta$ is the smaller the time nodes is. A consistent investor would buy later and vice versa.

*Remark* 3.5. In the industry, people sometimes look at moving averages and exponential moving averages. Some even look at Bolinger Bands or Relative Strength Index to determine whether the market is overbought or oversold. The idea is similar as the the proof indicates.

One can always observe a series of price actions heading one direction until the momentum dies out and price direction reverses.

## 3.5 Optimal Trading Frequency

Now we have a systematic way to describe our trading frequency and the understanding of an ideal price to pull the trigger. The next question is very intuitive. How do we make sure that our trading frequency matches the occurrences of "good" prices?

The idea is the following. We can observe prices going up or down from the dataset. From the price, we calculate the standard deviation (by taking first order derivatives). Each investor has some arbitrary coefficient, $h$, to trade, which is multiplied to the standard deviation. For example, for a mean standard deviation $\bar{\sigma}$, an institutional investor would be heavier and trade less and he could have a coefficient of 2. That is, he would not buy/sell unless the security is at a price level that gives him standard deviation of twice more than the mean standard deviation $2 \times \bar{\sigma}$. As an opposite example, an individual trader would be lighter and trade more and he could have a coefficient of 1.6. That is, if the price gives him a standard deviation of 1.6 higher or lower than the mean, $1.6 \times \bar{\sigma}$, he would pull the trigger to buy/sell. How do we know a person is being consistent and rational with his ideal trading frequency?

The answer is to let the trading frequency matches the probability of the appearance of the optimal trading price, which leads to the following theorem.

**Theorem 3.6.** *A trader has optimum at* $\sigma_{t,n} = \sigma_{t+\triangle,n}$ *and he should set his trading coefficient* $h$ *to be greater than* $\frac{\sigma'_{t,n}}{\sigma_{t',n'}}$.

This theorem is to show the audience a risk can be a good indicator of your trading frequency coefficient. A trader can observe price action and calculate some standard deviation, $\sigma_{t,n}$, at the beginning. As time moves on he observes more and more prices; and he can calculate an updated standard deviation, $\sigma'_{t,n}$. The theorem says one should trade only when he observes an extremely large updated standard deviation. In fact, he wants to observe a standard deviation that is greater than the standard deviation calculated by all the dataset in the past. Reader can see the proof in Appendix.

*Remark* 3.7. Majority of hedge funds and trading company conduct research on large data set. One week or even one more month of data would not affect the overall standard deviation by a lot. One needs to keep in mind that this approach may seem to have little difference day by day in practice.

The observed new standard deviation is a indicator that the risk is changing at that time node in the market. This can only happen with a lot of volume going on at that time node in the industry. This is usually due to some large counterparty unwinding his or her positions. This is usually a signal to get in or out of positions. This activity usually comes with significant volume.

# 4 Conclusion

In this paper we present a method to pick a critical price by using moving averages and investor trading frequency.

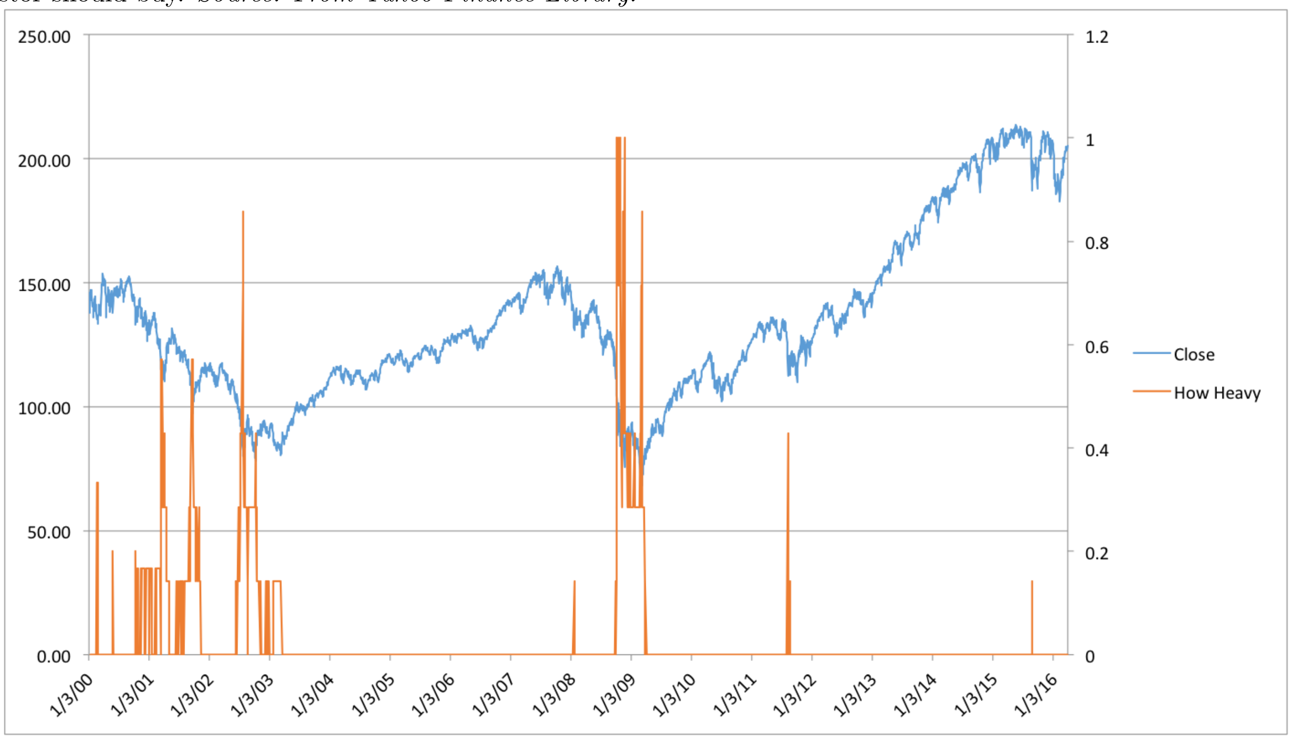
We assume price follows a random walk and we assume sufficient liquidities for to make a market. We provide a model such that an investor can find a critical price to be local minimum or local maximum to act on.

The essence of this paper rests in the goal of getting rid of the ups and downs of the equity market. We present this model hoping to help equity market to be less volatile.

## 4.1 Discussion of Applications

In practice, this model definitely can pick up some good price ranges. In Figure 5, we have shown the daily price plot for S&P 500 from January 2000 to January 2016 from Yahoo Data Library. We calculated the moving averages on 10-, 20-, 30-, 50-, 100-, 200-, and 300-day periods. For less than 10% trading frequency, we calculate the buy signal for each price-to-moving averages, which are presented as spikes on x-axis. The more price-to-moving averages based on different time periods satisfy the condition the higher the spikes are. The higher the spikes are the heavier an investor should buy. We observe that the spikes pretty much covered both financial crisis (2001 and 2008) and the model successfully tells us, by using only past data at each time node, that when to start buying and when to buy heavy.

Figure 5: The graph shows S&P 500 from January 2000 to January 2016. The model takes 10-, 20-, 30-, 50-, 100-, 200-, 300-day to be moving average periods and calculate the price-to-moving average ratio by investor trading frequency to be less than 10%. The spikes from x-axis are buy signal calculated based on critical price level. The higher the spikes are the heavier an investor should buy. *Source: From Yahoo Finance Library.*



# 5 Appendix

## 5.1 Proof of Theorem 2.1

*Proof:* Price $P_t$, a discrete-time stochastic variable, is adapted to $\mathbb{F}$, that is, $P_t$ is $F_t$-measurable. Moreover, $\mathbb{E}|P_t| < \infty$. Last, $E(P_{t+1}|F_t) = P_t$, that is, the expected price of the next time node is independent as the expected price of the current time node. We have seen empirical evidence from Malkiel (2007) [4].

We can also check for first order derivative of price $r_t = \frac{\partial P}{\partial t}$. We see that $r_t$ is $F_t$-measurable. Also $\mathbb{E}|r_t| < \infty$. Last, $E(r_{t+1}|F_t) = r_t$.

Q.E.D.

## 5.2 Proof of Theorem 2.2

*Proof:* We start with (ii) and then prove (i). Assume that for a given $t$ and $s$, we can write $\exists k > 0, t = s + k$. Since $\mathbb{F} = \{F_n, n = 0, 1, ...\}$ is a filtration, we have the following by tower property

$\mathbb{E}(r_t|F_s) = \mathbb{E}(\mathbb{E}(r_t|F_{t-1})|F_s) = \mathbb{E}(r_{t-1}|F_s)$

$= \mathbb{E}(\mathbb{E}(r_{t-1}|F_{t-2})|F_s) = \mathbb{E}(r_{t-2}|F_s)$

$= ...$

$= \mathbb{E}(\mathbb{E}(r_{t-(k-2)}|F_{t-(t-1)})|F_s) = \mathbb{E}(r_{t-(k-1)}|F_s)$

$$= \mathbb{E}(r_{s+1}|F_s) = r_s$$

Hence, this completes proof for (ii). For (i), we take $s = 0$ and (ii) gives us $\mathbb{E}(r_t|F_0) = r_0$. Then the expectations will be

$$\mathbb{E}(r_t) = \mathbb{E}(\mathbb{E}(r_t|F_0)) = \mathbb{E}(r_0)$$

which completes the proof.

Q.E.D.

## 5.3 Proof of Theorem 2.4

*Proof:* We have price-to-moving average to be the following. For $n$ periods, we have

$$P_t - SMA_n = \frac{P_t}{\frac{1}{n}\sum_{t=0}^{n} P_t} - 1$$

We want to know what time is the price the lowest/highest. Thus, we want to take derivative in respect with time $t$. Then set the first order derivative to be zero and we have

$$\frac{\partial(P_t - SMA_n)}{\partial t} = lim_{\triangle\to 0}\frac{(\frac{P_{t+\triangle}}{\frac{1}{n}\sum_{t=0}^{n} P_{t+\triangle}} - 1) - (\frac{P_t}{\frac{1}{n}\sum_{t=0}^{n} P_t} - 1)}{t} = 0$$

$$\Rightarrow lim_{\triangle\to 0}\frac{\frac{P_{t+\triangle}}{\frac{1}{n}\sum_{t=0}^{n} P_{t+\triangle}} - \frac{P_t}{\frac{1}{n}\sum_{t=0}^{n} P_t}}{t} = 0$$

$$\Rightarrow lim_{\triangle\to 0}\left(\frac{1}{t}\right)\left(\frac{P_{t+\triangle}}{\frac{1}{n}\sum_{t=0}^{n} P_{t+\triangle}} - \frac{P_t}{\frac{1}{n}\sum_{t=0}^{n} P_t}\right) = 0$$

$$\Rightarrow lim_{\triangle\to 0}\left(\frac{n}{t}\right)\left(\frac{P_{t+\triangle}}{\sum_{t=0}^{n}P_{t+\triangle}}-\frac{P_t}{\sum_{t=0}^{n}P_t}\right)=0$$

$$\Rightarrow lim_{\triangle\to 0}\left(\frac{n}{t}\right)$$

$$\left((P_{t+\triangle})\times(\sum_{t=0}^{n}P_t)-(P_t)\times(\sum_{t=0}^{n}P_{t+\triangle})\right)\times(\sum_{t=0}^{n}P_t\times\sum_{t=0}^{n}P_{t+\triangle})^{-1}=0$$

Then we need

$$\Rightarrow\left((P_{t+\triangle})\times(\sum_{t=0}^{n}P_t)-(P_t)\times(\sum_{t=0}^{n}P_{t+\triangle})\right)=0$$

Since $P_{t+\triangle}=P_t$ when $\triangle\to 0$ (the price this second is very close to the price next second), we can factor out one of them

$$\Rightarrow(P_{t+\triangle})\times\left((\sum_{t=0}^{n}P_t)-(\sum_{t=0}^{n}P_{t+\triangle})\right)=0$$

$$\Rightarrow(\sum_{t=0}^{n}P_t)-(\sum_{t=0}^{n}P_{t+\triangle})=0$$

Hence, when $(\sum_{t=0}^{n}P_t)=(\sum_{t=0}^{n}P_{t+\triangle})$, we have the optimal price in the market that has the first order derivative to be zero (or approximately zero), which shows that this price is a local minimum/maximum (a critical price or an actionable price).

Q.E.D.

## 5.4 Proof of Theorem 3.5

*Proof:* At the optimal trading price, we have $(\sum_{t=0}^{n}P_t)=(\sum_{t=0}^{n}P_{t+\triangle})$. For each time node $t$ and $t+\triangle$, we have

$$\sigma_{t,n}=\sqrt{\frac{1}{n}\sum_{i=1}^{n}((p_t-SMA_i)-\frac{1}{n}\sum_{i=1}^{n}(p_t-SMA_i))^2}$$

and

$$\sigma_{t+\triangle,n}=\sqrt{\frac{1}{n}\sum_{i=1}^{n}((p_{t+\triangle}-SMA_i)-\frac{1}{n}\sum_{i=1}^{n}(p_{t+\triangle}-SMA_i))^2}$$

Since $(\sum_{t=0}^{n}P_t)=(\sum_{t=0}^{n}P_{t+\triangle})$, then we have $\sigma_{t,n}=\sigma_{t',n'}$. Then for a particular time node $t$ and the next unit of time node $t+\triangle$,

$$\delta_t=Pr(\sigma'_{t,n}<h\sigma_{t,n})=\frac{\sum_{(t',n')}\mathbb{I}_{(t',n')}}{\sum_{(t,n)}\mathbb{I}_{(t,n)}},$$

we have $\mathbb{I}_{(t',n')}$ remains the same while $\mathbb{I}_{(t,n)}$ counts one more event. That is, we have $h'\sigma_{t',n'}=h'\sigma_{t'+\triangle,n'},\forall h'\in\mathbb{R}^+$. Note that here $h'$ is a different investor coefficient as the definition. Recall the definition that

$$\mathbb{I}_{(t',n')}=\begin{cases}1, \; when \; h\sigma_{t',n'}>\sigma'_{t,n}\\ 0, \; when \; h\sigma_{t',n'}\le\sigma'_{t,n}\end{cases},\ (sign\ changes\ for\ sell\ side)$$

which has an initial investor coefficient $h$. The initial coefficient is defined in the model. However, as time moves on and investor observes different return and standard deviation, he will have another coefficient, $h'$. You simply want to adjust trading frequency $h'$ to match the condition $(\sum_{t=0}^{n} P_t) = (\sum_{t=0}^{n} P_{t+\triangle})$.

That is, a trader has original coefficient $h$, and he wants to set $h$ to $h'$. When the indicator is 1, he trades. The indicator is 1 if standard deviation gets above or below a certain coefficient $h$ multiplied to the standard deviation. Now the optimal standard deviation is when $\sigma_{t,n} = \sigma_{t+\triangle,n}$. Then we have the following

$$h\sigma_{t',n'} > \sigma'_{t,n}$$

$$\Rightarrow h > \frac{\sigma'_{t,n}}{\sigma_{t',n'}},$$

yet since optimum occurs at $\sigma_{t,n} = \sigma_{t+\triangle,n}$, then

$$\Rightarrow \frac{\sigma_{t,n}}{dt} = lim_{\triangle\to\infty} \frac{\sigma_{t+\triangle,n} - \sigma_{t,n}}{\triangle}$$

$$\Rightarrow \frac{\sigma_{t,n}}{dt} = 0 \; as \; \triangle \to \infty.$$

Hence, we conclude a trader has optimum at $\sigma_{t,n} = \sigma_{t+\triangle,n}$ and he should set his trading coefficient $h$ to be greater than $\frac{\sigma'_{t,n}}{\sigma_{t',n'}}$.

Q.E.D.

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